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HOPF ALGEBRAS WITH RIGID DUALIZING COMPLEXES

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ABSTRACT

Let H be a Hopf algebra over a base field. If H has an $\mathbb{N}$-filtration such that the associated graded ring is connected graded noetherian and has enough normal elements, then H is Gorenstein. This gives a partial solution to a question of Brown and Brown–Goodearl. As a consequence, every quotient Hopf algebra of a generic quantized coordinate ring of a connected semisimple Lie group is Auslander–Gorenstein and Cohen–Macaulay. The last statement answers a question of Goodearl–Zhang.

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0. Introduction

In 1969, Larson–Sweedler [LS] proved that every finite dimensional Hopf algebra over a base field k is Frobenius; consequently, it has injective dimension zero. This result is extremely useful and fundamental in the study of finite dimensional Hopf algebras. It is natural to ask if a version of Larson–Sweedler's result holds for infinite dimensional noetherian Hopf algebras. In 1997, Brown–Goodearl [BG1, Subsection 1.15] and Brown [Br, Subsection 3.1] asked

Question 0.1: If H is a noetherian Hopf algebra, does H have finite injective dimension (or equivalently, is H Gorenstein)?

Since the results obtained in [BG1, Br], several partial solutions have been found. For example, Wu–Zhang answered this question affirmatively when H is PI [WZ2, Theorem 0.1] and when H is $\mathbb{N}$-graded with balanced dualizing complex [WZ3, Theorem 1]; Goodearl–Zhang solved this question for another class of Hopf algebras in [GZ, Theorem 0.2]. The dualizing complex was first used for this question in [WZ1, Corollary 0.4], where the question was answered for noetherian Hopf algebras that are finite over their centers. A few years ago Yekutieli suggested that the rigid dualizing complex should be useful for solving this question generally. One of our main results is to generalize some of these known partial results using rigid dualizing complexes (see Theorems 0.4 and 2.3).

Recent research in ring theory and noncommutative algebraic geometry suggests that we need a better understanding of certain homological properties for several classes of algebras, including quantum groups. For example, Goodearl and Zhang asked the following question [GZ, Question 0.3].

Question 0.2: Let G be a connected semisimple Lie group over $\mathbb{C}$, and I a Hopf ideal of the standard generic quantized coordinate ring $\mathcal{O}_q(G)$. Is the Hopf algebra $\mathcal{O}_q(G)/I$ Auslander–Gorenstein and Cohen–Macaulay?

The definitions of the Auslander–Gorenstein property and the Cohen–Macaulay property will be reviewed in Section 1. Many quantized algebras (including $\mathcal{O}_q(G)$ in the above question) have a good filtration which helps to build up rigid dualizing complexes.

Hypothesis 0.3: We say an algebra A satisfies **filtration hypothesis** if A has an exhaustive ascending $\mathbb{N}$-filtration such that the associated graded ring $\operatorname{gr} A$

is connected graded noetherian with **enough normal elements**, which means that every non-simple graded prime factor ring of $\operatorname{gr} A$ contains a homogeneous normal element of positive degree.

We use rigid dualizing complexes to prove the following results.

Theorem 0.4 (Corollary 2.6): *Let H be a noetherian Hopf algebra. Suppose H satisfies the filtration hypothesis. Then H is Gorenstein, and the following hold.*

(a) *H is Artin–Schelter Gorenstein;*
(b) *the rigid dualizing complex over H is $R = {}^{\nu}H^1[n]$ where $n = \operatorname{injdim} H = \operatorname{GKdim} H$ and ν is some algebra automorphism of H;*
(c) *H is Auslander–Gorenstein and Cohen–Macaulay;*
(d) *$\operatorname{GKdim} H/\mathfrak{p} = \operatorname{GKdim} H$ for every minimal prime ideal $\mathfrak{p} \subset H$;*
(e) *H has a quasi-Frobenius artinian ring of fractions;*
(f) *the antipode of H is bijective;*
(g) *if H has finite global dimension, then H is semiprime.*

Theorem 0.5: *Let G be a connected semisimple Lie group over $\mathbb{C}$, and I a Hopf ideal of the standard generic quantized coordinate ring $\mathcal{O}_q(G)$. Then $\mathcal{O}_q(G)/I$ is Auslander–Gorenstein and Cohen–Macaulay. Further, assertions (a), (b), (d), (e), (f) in Theorem 0.4 hold for $H := \mathcal{O}_q(G)/I$.*

Theorem 0.5 answers Question 0.2 affirmatively. Theorem 0.4 generalizes the first part of [GZ, Theorem 0.2] and answers Question 0.1 for a special case. The filtration hypothesis is mild since many quantum groups and quantum algebras (see for example [GL]) satisfy this hypothesis. Following ideas in the proof of [BG2, Theorem 2.6] one can show that if the R-matrix flips a full flag of subspaces of the generating vector space, then the quantum symmetric algebra (and then the corresponding quantum group in most cases) obtained by the FRT-construction satisfies the filtration hypothesis (see [BG2] for details).

The proof of Theorem 0.4 (and Theorem 0.5) is fairly easy and uses only basic properties of rigid dualizing complexes developed by Van den Bergh [VdB] and Yekutieli–Zhang [YZ1]-[YZ4]. Our contribution is to find a formula which serves as a bridge between properties of the rigid dualizing complex and the Gorenstein property of a Hopf algebra (Theorems 2.3(c) and 2.4(b)). This formula states that, under certain conditions such as the filtration hypothesis, the isomorphism

$$\mathrm{RHom}_{A^{\mathsf{e}}}(A, A \otimes R^{\mathsf{op}}) \cong A$$

holds for the rigid dualizing complex R over A (Lemma 1.8). This paper provides a surprising application of the rigid dualizing complex to the study of noetherian Hopf algebras. Our method is not complicated, it could be modified to cover a larger class of Hopf algebras.

Using similar ideas some other results are proved for (semi)prime noetherian Hopf algebras. The following is analogous to [YZ5, Theorem 0.2]. The terminology used next is defined in Section 3.

Theorem 0.6 (Theorem 3.2): *Let H be a prime noetherian Artin–Schelter regular Hopf algebra of global dimension d. Let Q be the simple artinian Goldie quotient ring of H. Then Q is rigid and smooth of homological transcendence degree d (in the sense of* [YZ5]*).*

Theorem 0.6 generalizes some results of Stafford [St] and Yekutieli–Zhang [YZ5]. A partial converse of Theorem 0.6 is stated in Proposition 3.6.

1. Homological algebra preparations

Throughout, let k be a commutative base field. All vector spaces are over k; in particular, an algebra or a ring means a k-algebra, and the unmarked tensor product $\otimes$ means $\otimes_k$. In a small part of the paper we are working with the quantized coordinate ring $\mathcal{O}_q(G)$ and its factor Hopf algebras, in which case we need to assume that k is the field of complex numbers $\mathbb{C}$ as in [GZ].

Usually we work with left modules. The category of left A-modules is denoted by A-Mod and the category of right A-modules is denoted by Mod-A. Let A^{op} denote the opposite ring of A. A right A-module can be viewed as a left A^{op}-module. Let A^{e} denote the ring $A \otimes A^{\mathsf{op}}$. An A-bimodule is sometimes identified with a left A^{e}-module.

We refer to Montgomery's book [Mo] for basic definitions related to Hopf algebras. Let H be a Hopf algebra over k. Usually k denotes the trivial H-bimodule $H/\ker\epsilon$, where $\epsilon : H \to k$ is the counit of H. We refer to Krause–Lenagan's book [KL] for the basics about Gelfand–Kirillov dimension.

In this section, we recall several definitions related to homological properties and rigid dualizing complexes of noncommutative noetherian rings.

Definition 1.1: A noetherian algebra A is called **Gorenstein** if it has finite injective dimension on both sides.

Definition 1.2: A Hopf algebra H is called **Artin–Schelter Gorenstein** (or **AS–Gorenstein**) if

(AS1) $\operatorname{injdim}{}_H H = d < \infty$;
(AS2) $\dim_k \operatorname{Ext}^d_H({}_H k, {}_H H) = 1$, $\operatorname{Ext}^i_H({}_H k, {}_H H) = 0$ for all $i \neq d$;
(AS3) the right H-module versions of the conditions (AS1,AS2) hold.

We say H is **Artin–Schelter regular** (or **AS-regular**) if it is AS–Gorenstein and it has finite global dimension.

To define the Auslander–Gorenstein and Cohen–Macaulay conditions we need more invariants of modules.

Definition 1.3: Let A be a noetherian Gorenstein algebra.

(1) Let M be a finitely generated left (or right) A-module, the **grade** or the j**-number** of M with respect to A is defined to be
$$j(M) := \inf\{n :\ \operatorname{Ext}^n_A(M, A) \neq 0\}.$$
(2) The ring A is called **Auslander–Gorenstein** if it satisfies the **Auslander condition**:

For every finitely generated left (respectively, right) A-module M and every nonnegative integer q, one has $j(N) \geq q$ for every finitely generated right (respectively, left) A-submodule $N \subseteq \operatorname{Ext}^q_A(M, A)$.

(3) The ring A is called **Auslander regular** if it is Auslander–Gorenstein and it has finite global dimension.
(4) Suppose A has finite Gelfand–Kirillov dimension (denoted by GKdim). We say that A is **Cohen–Macaulay** (**with respect to** GKdim) if
$$j(M) + \operatorname{GKdim} M = \operatorname{GKdim} A$$
for every nonzero finitely generated left or right A-module M.

The concepts in Definition 1.3 (1,2,4) can also be defined for (rigid) dualizing complexes, which will be reviewed next. Let $\mathsf{D}(A\text{-}\mathsf{Mod})$ denote the derived category of left A-modules. Let $\mathsf{D}^+(A\text{-}\mathsf{Mod})$ (respectively, $\mathsf{D}^-(A\text{-}\mathsf{Mod})$, $\mathsf{D}^{\mathrm{b}}(A\text{-}\mathsf{Mod})$) denote the bounded below (respectively, bounded above, bounded) derived category of left A-modules. Let X be a nonzero bounded complex in

$\mathsf{D}^{\mathrm{b}}(A\text{-}\mathsf{Mod})$. The injective dimension of X is defined to be

(E1.3.1) $$\operatorname{injdim} X = \sup\{i : \operatorname{Ext}^i_A(M, X) \neq 0, \text{ for some } M \in A\text{-}\mathsf{Mod}\}.$$

If A is noetherian, then we need only use finitely generated A-modules M in (E1.3.1). Let I be a minimal injective resolution of X. Then $\operatorname{injdim} X = \max\{i : I^i \neq 0\}$. The projective dimension of X is defined to be

(E1.3.2) $$\operatorname{projdim} X = \sup\{i : \operatorname{Ext}^i_A(X, M) \neq 0, \text{ for some } M \in A\text{-}\mathsf{Mod}\}.$$

If A is noetherian and X is a compact object in $\mathsf{D}(A\text{-}\mathsf{Mod})$, then X has finite projective dimension and $\operatorname{projdim} X = \max\{i : \operatorname{Ext}^i_A(X, A) \neq 0\}$. The following definition is due to Yekutieli [Ye].

Definition 1.4: Let A be a noetherian algebra. A complex $R \in \mathsf{D}^{\mathrm{b}}(A^{\mathsf{e}}\text{-}\mathsf{Mod})$ is called a **dualizing complex** over A if it satisfies the following conditions:

(a) R has finite injective dimension over A and over A^{op}, respectively;
(b) for every i, $\mathrm{H}^i R$ is finitely generated over A and over A^{op}, respectively;
(c) the canonical morphisms $A \to \operatorname{RHom}_A(R, R)$ and $A \to \operatorname{RHom}_{A^{\mathsf{op}}}(R, R)$ are isomorphisms in $\mathsf{D}(A^{\mathsf{e}}\text{-}\mathsf{Mod})$.

For any complex X and any integer n, the nth complex shift is denoted by $X[n]$. If A is noetherian and Gorenstein, then any complex shift $A[n]$ of the A-bimodule A is a dualizing complex over A. Let X be a complex of A-bimodules. We use ${}_AX$ and X_A to denote the same complex with the induced left and right A-module structure respectively.

Let R be a complex of A^{e}-modules (or a complex of A-bimodules). Let R^{op} denote the "opposite complex" of R which is defined as follows: as a complex of k-modules $R^{\mathsf{op}} = R$, and the left and right A^{op}-module actions on R^{op} are given by

$$a \cdot r := ra \quad \text{and} \quad r \cdot b := br,$$

respectively, for all $a, b \in A^{\mathsf{op}}(= A)$ and $r \in R^{\mathsf{op}}(= R)$. If $R \in \mathsf{D}(A^{\mathsf{e}}\text{-}\mathsf{Mod})$, then $R^{\mathsf{op}} \in \mathsf{D}^{\mathrm{b}}((A^{\mathsf{op}})^{\mathsf{e}}\text{-}\mathsf{Mod})$. The flip map

$$\phi : (A^{\mathsf{e}})^{\mathsf{op}} = (A^{\mathsf{op}})^{\mathsf{e}} = A^{\mathsf{op}} \otimes A \longrightarrow A \otimes A^{\mathsf{op}} = A^{\mathsf{e}}$$

is an algebra isomorphism. Hence there is a natural isomorphism $\mathsf{D}^{\mathrm{b}}(A^{\mathsf{e}}\text{-}\mathsf{Mod}) \cong \mathsf{D}^{\mathrm{b}}((A^{\mathsf{op}})^{\mathsf{e}}\text{-}\mathsf{Mod})$. It is obvious that R is a dualizing complex over A if and only if R^{op} is a dualizing complex over A^{op}. The following definition is due to Van den Bergh [VdB].

Definition 1.5: Let A be a noetherian algebra. A dualizing complex R over A is called **rigid** if there is an isomorphism

$$R \cong \mathrm{RHom}_{A^{\mathsf{e}}}(A, R \otimes R^{\mathsf{op}})$$

in $\mathsf{D}(\mathsf{Mod}\text{-}A^{\mathsf{e}})$. Here the left A^{e}-module structure of $R \otimes R^{\mathsf{op}}$ comes from the left A-module structure of R and the left A^{op}-module structure of R^{op}.

Let R be a dualizing complex over A (not necessarily rigid), and let M be a finitely generated left (or right) A-module. The **grade** of M **with respect to** R is defined to be

$$j_R(M) := \inf\{n : \mathrm{Ext}^n_A(M, R) \neq 0\}.$$

We say that R is **Auslander** if:

> For every finitely generated left (respectively, right) A-module M, every integer q, and every finitely generated right (respectively, left) A-submodule $N \subseteq \mathrm{Ext}^q_A(M, R)$, one has $j_R(N) \geq q$.

We say that R is **Cohen–Macaulay** if

$$j_R(M) + \mathrm{GKdim}\, M = 0$$

for all nonzero finitely generated left and right A-modules M.

Let M be an A-bimodule and let ν, τ be algebra automorphisms of A. The twisted bimodule ${}^{\nu}M^{\tau}$ is an A-bimodule defined by

$$a \cdot m \cdot b \;=\; \nu(a) m \tau(b)$$

for all $a, b \in A$ and all $m \in {}^{\nu}M^{\tau} (= M)$. If τ is the identity map of A, ${}^{\nu}M^{\tau}$ is written as ${}^{\nu}M^{1}$.

The connection between the Auslander and Cohen–Macaulay properties just defined and the ones defined in Definition 1.3 are the following, which is well-known. See [YZ2, Proposition 3.4] for similar statements.

Lemma 1.6: *Let A be a noetherian algebra. Let n be any integer and ν be any algebra automorphism of A.*

(a) *The ring A is Gorenstein if and only if ${}^{\nu}A^{1}[n]$ is a dualizing complex.*
(b) *Suppose A is Gorenstein. Then the ring A is Auslander–Gorenstein if and only if the dualizing complex ${}^{\nu}A^{1}[n]$ is Auslander.*

(c) *Suppose A is Gorenstein with* $\operatorname{GKdim} A = d < \infty$. *Then the ring A is Cohen–Macaulay if and only if the dualizing complex* ${}^{\nu}A^1[d]$ *is Cohen–Macaulay.*

In some cases, the rigid dualizing complex will have the form of $R = {}^{\nu}A^1[n]$ for some algebra automorphism ν of A (see, for example, Theorem 2.4(a)).

Lemma 1.7: *Suppose A satisfies filtration hypothesis 0.3. Then*

(a) *there is a rigid dualizing complex R over A;*
(b) *R is Auslander and Cohen–Macaulay;*
(c) *$A^{\mathsf{e}} := A \otimes A^{\mathsf{op}}$ is noetherian;*
(d) *$R \otimes R^{\mathsf{op}}$ is a (rigid) dualizing complex over A^{e}.*

Proof. (a), (b) [YZ1, Corollary 6.9(iii)].

(c) It follows from [ArSZ, Propositions 4.3, 4.9 and 4.10] that $A \otimes B$ is noetherian for any noetherian algebra B. Hence $A \otimes A^{\mathsf{op}}$ is noetherian.

(d) Suppose A and B are two algebras satisfying filtration hypothesis 0.3. Let R_A and R_B be the rigid dualizing complexes over A and B, respectively. [YZ4, Theorem 8.5] implies that $R_A \otimes R_B$ is the rigid dualizing complex over $A \otimes B$. The hypotheses in [YZ4, Theorem 8.5] are slightly different from ours, but the proof of [YZ4, Theorem 8.5] works under filtration hypothesis 0.3. The assertion follows by taking $B = A^{\mathsf{op}}$. ∎

Lemma 1.8: *Let A be an algebra. Suppose that*

(i) *A^{e} is noetherian,*
(ii) *R is the rigid dualizing complex over A, and*
(iii) *$R \otimes R^{\mathsf{op}}$ has finite injective dimension on both sides.*

Then

(a) *$R \otimes R^{\mathsf{op}}$ is a dualizing complex over A^{e}, and*
(b) $\operatorname{RHom}_{A^{\mathsf{e}}}(A, A \otimes R^{\mathsf{op}}) \cong A$ *in* $\mathsf{D}(\mathsf{Mod}\text{-}A^{\mathsf{e}})$.

Proof. (a) Definition 1.4(a) is (iii) and Definition 1.4(b,c) follows from a direct computation using the Künneth formula (see Lemma 1.9 below) and the fact that R (respectively, R^{op}) is a dualizing complex over A (respectively, over A^{op}).

(b) Since $R \otimes R^{\mathsf{op}}$ is a dualizing complex over A^{e}, we have

$$\begin{aligned}
&\mathrm{RHom}_{A^{\mathsf{e}}}(A, A \otimes R^{\mathsf{op}}) \\
&\quad\cong \mathrm{RHom}_{(A^{\mathsf{e}})^{\mathsf{op}}}(\mathrm{RHom}_{A^{\mathsf{e}}}(A \otimes R^{\mathsf{op}}, R \otimes R^{\mathsf{op}}), \mathrm{RHom}_{A^{\mathsf{e}}}(A, R \otimes R^{\mathsf{op}})) \\
&\quad\cong \mathrm{RHom}_{(A^{\mathsf{e}})^{\mathsf{op}}}(R \otimes A, \mathrm{RHom}_{A^{\mathsf{e}}}(A, R \otimes R^{\mathsf{op}})) \\
&\quad\cong \mathrm{RHom}_{(A^{\mathsf{e}})^{\mathsf{op}}}(R \otimes A, R)
\end{aligned}$$

where the second last isomorphism follows from the isomorphism

$$\mathrm{RHom}_{A^{\mathsf{op}}}(R^{\mathsf{op}}, R^{\mathsf{op}}) \cong {}_{A^{\mathsf{op}}}A^{\mathsf{op}} \cong A_A,$$

and the last isomorphism follows from the rigidity isomorphism of R given in Definition 1.5. By the Hom and $\otimes$ adjunction,

$$\begin{aligned}
\mathrm{RHom}_{(A^{\mathsf{e}})^{\mathsf{op}}}(R \otimes A, R) &\cong \mathrm{RHom}_{A^{\mathsf{op}}}(R, \mathrm{RHom}_A(A, R)) \\
&\cong \mathrm{RHom}_{A^{\mathsf{op}}}(R, R) \cong A. \qquad \blacksquare
\end{aligned}$$

The following special case of the Künneth formula is of course well-known.

Lemma 1.9: *Let X and Y be two complexes of k-vector spaces.*

(a) *For every n, $\mathrm{H}^n(X \otimes Y) = \bigoplus_{i\in\mathbb{Z}} \mathrm{H}^i(X) \otimes \mathrm{H}^{n-i}(Y)$.*

(b) *If $\mathrm{H}^i(X \otimes Y) = 0$ for all $i \neq 0$, then there is an n such that $X \cong M[n]$ and $Y \cong N[-n]$ for some k-vector spaces M and N, and $\mathrm{H}^0(X \otimes Y) \cong M \otimes N$.*

2. Gorenstein property of Hopf algebras

Throughout this section let H be a noetherian Hopf algebra. We need to recall the left adjoint action

$$L : H^{\mathsf{e}}\text{-}\mathsf{Mod} \longrightarrow H\text{-}\mathsf{Mod}$$

which is defined in [Mo, Definition 3.4.1(1)] (and is also reviewed in [BZ, 2.2]). Let M be an H-bimodule (or equivalently a left $H \otimes H^{\mathsf{op}}$-module). Then $L(M)$ is a left H-module where the left H-action is defined by

$$h \cdot m = \sum h_1 m S(h_2)$$

for all $h \in H$ and all $m \in M$. If $f : M \to N$ is an H-bimodule homomorphism, then $L(f) : L(M) \to L(N)$ is a left H-module homomorphism, which is the same as f when we consider $L(M) = M$ and $L(N) = N$ as k-vector spaces. Hence L is in fact a functor. We list some easy facts.

Lemma 2.1 ([BZ, Lemmas 2.1 and 2.2]): *Let A be any algebra. Let L be the left adjoint functor of H defined as above.*

(a) *L is an exact functor.*
(b) *L preserves projectives and injectives respectively.*
(c) *Let N be an $H^{\mathsf{op}} \otimes A^{\mathsf{op}}$-module. Then $L(H \otimes N)$ with induced A^{op}-action obtained from N is isomorphic to ${}_H H \otimes {}_k N_A$ as $H \otimes A^{\mathsf{op}}$-modules. The isomorphism from $L(H \otimes N)$ to ${}_H H \otimes {}_k N_A$ is given by*

$$h \otimes n \mapsto \sum h_1 \otimes nS^2(h_2),$$

where $\Delta(h) = \sum h_1 \otimes h_2$, with inverse $h' \otimes n' \mapsto \sum h'_1 \otimes n'S(h'_2)$. As a consequence, $L(H \otimes N)$ is a free left H-module.

Proof. See [BZ, Lemma 2.2] for (a), (b). Part (c) is similar to [BZ, Lemma 2.1(b)] and [BZ, Lemma 2.2(c)]. ∎

Since L is exact and preserves the injectives, we can extend L to a functor

$$L : \mathsf{D}^+(H^{\mathsf{e}}\text{-}\mathsf{Mod}) \to \mathsf{D}^+(H\text{-}\mathsf{Mod})$$

without taking the right derived functor. In other words, the right derived functor of L is L itself induced from the level of chain complexes.

Lemma 2.2: *Let k be the trivial H-module.*

(a) *Let X be a bounded below complex of H-bimodules. Then there is a natural isomorphism*

$$\mathrm{Ext}^i_{H^{\mathsf{e}}}(H, X) \cong \mathrm{Ext}^i_H(k, L(X))$$

for all i. This isomorphism preserves any extra module structure from X.

(b) *Let Y be a bounded complex of $H^{\mathsf{op}} \otimes A^{\mathsf{op}}$-modules. Then there is a natural isomorphism of A^{op}-modules*

$$\mathrm{Ext}^i_{H^{\mathsf{e}}}(H, H \otimes Y) \cong \mathrm{H}^i(\mathrm{RHom}_H(k, {}_H H) \otimes Y) \cong \bigoplus_{n \in \mathbb{Z}} \mathrm{Ext}^{i-n}_H(k, H) \otimes \mathrm{H}^n(Y)$$

for all i.

Proof. (a) If X is just an H-bimodule, the assertion is [BZ, Lemma 2.4(b)]. The proof of [BZ, Lemma 2.4(b)] works for a general X.

(b) By part (a), we have $\operatorname{Ext}^i_{H^e}(H, H\otimes Y)\cong \operatorname{Ext}^i_H(k, L(H\otimes Y))$. By Lemma 2.1(c), ${}_HL(H\otimes Y)_A\cong {}_HH\otimes {}_kY_A$. Hence

$$\begin{aligned}\operatorname{Ext}^i_H(k, L(H\otimes Y)) &\cong \operatorname{Ext}^i_H(k, {}_HH\otimes Y) = \mathrm{H}^i(\mathrm{RHom}_H(k, {}_HH\otimes Y))\\ &\cong \mathrm{H}^i(\mathrm{RHom}_H(k, {}_HH)\otimes Y)\\ &\cong \bigoplus_{n\in\mathbb{Z}} \operatorname{Ext}^{i-n}_H(k,H)\otimes \mathrm{H}^n(Y)\end{aligned}$$

where the last isomorphism follows from the Künneth formula (Lemma 1.9(a)) and the second last isomorphism holds since H is noetherian and k is finitely generated over H. The assertion follows by combining these isomorphisms. ∎

We are now ready to prove one of our main results.

Theorem 2.3: *Let H be a noetherian Hopf algebra. Suppose*

(i) *there is a dualizing complex R over H such that*

$$\mathrm{RHom}_{H^e}(H, H\otimes R^{\mathsf{op}})\cong H \tag{E2.3.1}$$

in $\mathsf{D}(\mathsf{Mod}\text{-}H^{\mathsf{e}})$,

(ii) *either H has an anti-automorphism or condition (i) hold for H^{op}.*

Then the following statements hold.

(a) *There is an n such that $\operatorname{Ext}^n_H(k,H)=k$ and $\operatorname{Ext}^i_H(k,H)=0$ for all $i\neq n$.*

(b) *R has the form of ${}^{\nu}H^1[n]$ where ν is an algebra automorphism of H.*

(c) *H is Gorenstein.*

(d) *If* $\operatorname{injdim} R=0$, *then H is AS-Gorenstein and* $\operatorname{injdim} H=n$.

Note that if the Hopf algebra H satisfies condition (i), (ii), (iii) in Lemma 1.8, then (E2.3.1) holds for the rigid dualizing complex over H.

Proof of Theorem 2.3. (a), (b), (c) First we only assume hypothesis (i). By (E2.3.1)

$$\operatorname{Ext}^i_{H^e}(H, H\otimes R^{\mathsf{op}})\cong \begin{cases} H & i=0,\\ 0 & i\neq 0.\end{cases}$$

By Lemma 2.2(b),

$$\mathrm{H}^0(\mathrm{RHom}_H(k,H)\otimes R^{\mathsf{op}})\cong \operatorname{Ext}^0_{H^e}(H, H\otimes R^{\mathsf{op}})\cong H$$

as left H-modules and

$$\mathrm{H}^i(\mathrm{RHom}_H(k,H)\otimes R^{\mathsf{op}}) \cong \mathrm{Ext}^i_{H^{\mathsf{e}}}(H, H\otimes R^{\mathsf{op}}) = 0$$

for all $i \neq 0$. By the Künneth formula [Lemma 1.9(b)], there is an n such that $\mathrm{Ext}^n_H(k,H) = V \neq 0$ and $\mathrm{Ext}^i_H(k,H) = 0$ for all i and $R^{\mathsf{op}} = M[n]$ for an H-bimodule M such that $V \otimes M \cong H$ as left H-module. This implies that $w := \dim_k V$ is finite and $H \cong M^{\oplus w}$. Since ${}_H R^{\mathsf{op}} (= {}_H R)$ has finite injective dimension, so has M. Hence ${}_H H$ has finite injective dimension.

Now we use hypothesis (ii). If H has an anti-automorphism, H_H has finite injective dimension. If hypothesis (i) holds for H^{op} (or for the Hopf algebra $(H, m^{\mathsf{op}}, \Delta^{\mathsf{op}}, S, \epsilon)$), the above argument shows that H_H has finite injective dimension.

Under the hypothesis (ii), we also see that

$$\mathrm{Ext}^i_{H^{\mathsf{op}}}(k,H) \cong \begin{cases} W & i = n, \\ 0 & i \neq 0 \end{cases}$$

for some finite dimensional vector space W. By the proof of [BZ, Lemma 3.2], V and W are 1-dimensional over k. Note that the proof of [BZ, Lemma 3.2] works here even if n is not the injective dimension of H, and in this case H is not AS-Gorenstein. Therefore (a) follows.

Since V is 1-dimensional, $M \cong H$ as a left H-module. This means that the dualizing complex R is isomorphic to $H[n]$ when restricted to the left-hand side. Since H is Gorenstein, R is a two-sided tilting complex over (H,H) by [YZ2, Lemma 5.2(2)]. The two-sided tilting complex R is invertible [YZ2, Definition 2.1] with inverse $R^\vee \cong \mathrm{RHom}_H(R,H)$. Since ${}_H R \cong {}_H H[n]$, $R^\vee \cong M^\vee[-n]$ for some H-bimodule $M^\vee$. The invertibility of R implies that M is invertible with an inverse $M^\vee$. Since ${}_H M \cong {}_H H$, the Morita theory implies that $M_H \cong H_H$. Therefore there is an algebra automorphism ν such that $M \cong {}^\nu H^1$ as H-bimodules. Part (b) follows, and so is part (c).

(d) By part (b),

$$\operatorname{injdim} H = \operatorname{injdim} H[n] + n = \operatorname{injdim} R + n < \infty,$$

where n is defined in parts (a), (b). If $\operatorname{injdim} R = 0$, then $\operatorname{injdim} H = n$. Combining with part (a), (d) follows. ∎

Every noetherian AS-Gorenstein Hopf algebra has a rigid dualizing complex.

Theorem 2.4: *Let H be a noetherian AS-Gorenstein Hopf algebra with bijective antipode. Let d be the injective dimension of H.*

(a) [BZ, Theorem 0.2] *The rigid dualizing complex R over H is isomorphic to ${}^{\nu}H^1[d]$ where ν is an algebra automorphism of H.*

(b) *The isomorphism* (E2.3.1) *holds for the R in part (a).*

Proof. (b) Since $R \cong {}^{\nu}H^1[d]$, R is an invertible H-bimodule complex and the inverse R^{-1} is ${}^1H^{\nu}[-d]$. Hence

$$\begin{aligned}\mathrm{RHom}_{H^{\mathsf{e}}}(H, H \otimes R^{\mathsf{op}}) &\cong \mathrm{RHom}_{H^{\mathsf{e}}}(H, (R \otimes R^{\mathsf{op}}) \otimes_{H^{\mathsf{e}}} (R^{-1} \otimes H^{\mathsf{op}}))\\ &\cong \mathrm{RHom}_{H^{\mathsf{e}}}(H, (R \otimes R^{\mathsf{op}})) \otimes_{H^{\mathsf{e}}} (R^{-1} \otimes H^{\mathsf{op}})\\ &\cong R \otimes_{H^{\mathsf{e}}} (R^{-1} \otimes H^{\mathsf{op}}) \cong H \otimes_H R \otimes_H R^{-1}\\ &\cong H. \qquad \blacksquare\end{aligned}$$

The algebra automorphism ν in the above theorem is called the **Nakayama automorphism** of H which is described in [BZ, Theorem 0.3] as follows:

$$\nu = S^2\xi,$$

where S is the antipode of H and ξ is the left winding automorphism of H associated to the left homological integral of H. We do not need this formula in this paper.

The following lemma is well-known.

Lemma 2.5: *Let A be a Hopf algebra (or any algebra with a nonzero finite dimensional A-module) and let R be a dualizing complex over A. If R is Auslander and Cohen–Macaulay, then* $\operatorname{injdim} R = 0$ *and* $\operatorname{GKdim} A = -j_R(A) = -\min\{i : \mathrm{H}^i(R) \neq 0\}$.

Proof. Since $j_R(M) = -\operatorname{GKdim} M \leq 0$ for every nonzero A-submodule $M \subset \operatorname{Ext}^i_{A^{\mathsf{op}}}(N, R)$, the Auslander condition implies that $\operatorname{Ext}^i_{A^{\mathsf{op}}}(N, R) = 0$ for all $i > 0$ and for all finitely generated A^{op}-modules N. Thus $\operatorname{injdim} R_A \leq 0$. By symmetry, $\operatorname{injdim} {}_AR \leq 0$. Since A has a 1-dimensional module k, $j_R(k) = -\operatorname{GKdim} k = 0$, $\operatorname{injdim} {}_AR \geq 0$. Therefore, $\operatorname{injdim} {}_AR = \operatorname{injdim} R_A = 0$. The last formula follows from the definition of Cohen–Macaulay property of R. ∎

Now we are ready to prove Theorem 0.4.

Corollary 2.6: *Suppose that H is a Hopf algebra satisfying filtration Hypothesis 0.3. Then the following hold.*

(a) *H is Gorenstein and AS-Gorenstein.*
(b) *The rigid dualizing complex R over H is isomorphic to ${}^{\nu}H^1[n]$ where $n = \operatorname{injdim} H = \operatorname{GKdim} H$.*
(c) *H is Auslander–Gorenstein and Cohen–Macaulay.*
(d) $\operatorname{GKdim} H/\mathfrak{p} = \operatorname{GKdim} H$ *for every minimal prime ideal* $\mathfrak{p} \subset H$.
(e) *H has a quasi-Frobenius artinian ring of fractions.*
(f) *The antipode of H is bijective.*
(g) *If H has finite global dimension, then H is semiprime.*

Proof. By Lemmas 1.7 and 1.8, the rigid dualizing complex R over H exists and (E2.3.1) holds for R. Hence we have verified hypothesis (i) in Theorem 2.3. Hypothesis 0.3 is left-right symmetric. Therefore hypothesis (ii) in Theorem 2.3 also holds.

(a) By Lemma 1.7(a,b), the rigid dualizing complex R over H is Auslander and Cohen–Macaulay. By Lemma 2.5, $\operatorname{injdim} R = 0$. By Theorem 2.3(c,d), H is Gorenstein and AS-Gorenstein.

(b) By Theorem 2.3(b,d), R is isomorphic to ${}^{\nu}H^1[n]$ where $n = \operatorname{injdim} H$. By Lemma 2.5,

$$\operatorname{GKdim} H = -\min\{i : H^i(R) \neq 0\} = -(-n) = n.$$

(c) By Lemma 1.7(b), R is Auslander and Cohen–Macaulay. The assertion follows from Lemma 1.6(b), (c) and part (b).

(d,e) Follow from part (c) and [AjSZ, Theorem 6.1]. The hypothesis (*) in [AjSZ, Theorem 6.1] holds trivially since H is Cohen–Macaulay with respective to GK-dimension.

(f) This follows from [Sk, Theorem A(ii)] and part (e).

(g) Follows from part (c) and [AjSZ, Corollary 6.3]. ∎

Theorem 0.5 is an immediate consequence.

Proof of Theorem 0.5. By [GZ, Theorem 2.6], $\mathcal{O}_q(G)$ satisfies filtration Hypothesis 0.3. Hence any quotient $\mathcal{O}_q(G)/I$ satisfies filtration Hypothesis 0.3. The assertions follow from Corollary 2.6. ∎

3. Goldie quotient rings of (semi)prime Hopf algebras

Semisimple artinian Hopf algebras have been studied by many authors. Properties of general infinite dimensional Hopf algebras are less known. In this section we study some general ring-theoretic properties of semiprime noetherian Hopf algebras with finite global dimension. Note that, by [Sk, Corollary 1], every semiprime noetherian Hopf algebra has a bijective antipode. The results in this section are not related to the ones in the last section though some ideas in the proofs are similar. First we recall a result about noetherian PI Hopf algebras.

THEOREM 3.1: *Let H be a noetherian affine PI Hopf algebra of* GKdim d. *Then the following hold*

(a) *H is AS-Gorenstein, Auslander Gorenstein and Cohen–Macaulay of injective dimension d.*
(b) *If* $\operatorname{gldim} H < \infty$, *then H is a direct sum of prime rings of the same* GKdim *and the center of H is a direct sum of Krull domains of the same* GKdim, *and H is finite over its center and each prime direct summand is equal to its trace ring.*
(c) *Suppose* $\operatorname{gldim} H < \infty$. *Let $Q(H)$ be the Goldie quotient ring of H. Then $Q(H)$ is a direct sum of simple artinian algebras of the same transcendence degree.*

Proof. (a) This is [WZ2, Theorems 0.1 and 0.2].
(b) Follows from [SZ, Theorems 5.4 and 5.6].
(c) Clear from part (b). ∎

As we have seen from Corollary 2.6, a version of Theorem 3.1(a), (b) should hold for non-PI noetherian Hopf algebras. In this section we will prove a version of Theorem 3.1(c) for certain Hopf algebras. The classical transcendence degree is not defined for non-PI division algebras, but there is a replacement. Let D be a simple artinian algebra (over k). The **homological transcendence degree** of D [YZ5, Definition 1.1(a)] is defined to be

$$\operatorname{Htr} D = \operatorname{injdim}_{D^{\mathrm{e}}} D.$$

We refer to [YZ5] for the basic properties of Htr. A simple artinian algebra D is called **rigid** [YZ5, Definition 2.2, p. 115] if

$$\operatorname{Ext}^i_{D^{\mathrm{e}}}(D, D^{\mathrm{e}}) = \begin{cases} {}^{\nu}D^1 & i = \operatorname{Htr} D, \\ 0 & i \neq \operatorname{Htr} D, \end{cases}$$

where ν is an algebra automorphism of D. A simple artinian algebra D is called **smooth** if $\operatorname{gldim} D^{\mathsf{e}} < \infty$ (or equivalently $\operatorname{projdim}_{D^{\mathsf{e}}} D < \infty$) (see [YZ5, Definition 1.1(f) and Lemma 1.3(c)]).

THEOREM 3.2: *Let H be a semiprime noetherian AS-regular Hopf algebra of global dimension d. Let $Q(H)$ be the Goldie quotient ring of H. Then $Q(H)$ is a direct sum of rigid and smooth simple artinian algebra of the same homological transcendence degree d.*

Proof. Since $Q(H)$ is semisimple artinian, it has an algebra decomposition into simple artinian rings

$$Q(H) = \bigoplus_j D_j. \tag{E3.2.1}$$

Since H has finite global dimension, $\operatorname{projdim}_H k < \infty$ and $\operatorname{projdim}_{H^{\mathsf{op}}} k < \infty$. By the Künneth formula,

$$\operatorname{projdim}_{(H\otimes H^{\mathsf{op}})} k \leq \operatorname{projdim}_H k + \operatorname{projdim}_{H^{\mathsf{op}}} k < \infty.$$

Hence the Hopf algebra $H \otimes H^{\mathsf{op}}$ has finite global dimension by [LL, Corollary 2.4]. Therefore its localization $Q(H) \otimes Q(H^{\mathsf{op}})$ has finite global dimension. Since $D_j \otimes D_j^{\mathsf{op}}$ is a direct summand of $Q(H) \otimes Q(H^{\mathsf{op}})$, $D_j \otimes D_j^{\mathsf{op}}$ has finite global dimension. This implies that D_j is smooth for every j.

Since H is AS-regular, Theorem 2.4(a) says that H has a rigid dualizing complex of the form ${}^{\nu}H^1[d]$ where $d = \operatorname{injdim} H = \operatorname{gldim} H$. By [BZ, Lemma 5.2(c)], H is a compact object in $\mathsf{D}(H^{\mathsf{e}}\text{-}\mathsf{Mod})$. Then

$$\begin{aligned}
&\operatorname{RHom}_{H^{\mathsf{e}}}(H, R \otimes R^{\mathsf{op}}) \otimes_{H^{\mathsf{e}}} (Q(H) \otimes Q(H^{\mathsf{op}}))\\
&\qquad \cong \operatorname{RHom}_{H^{\mathsf{e}}}(H, (R \otimes R^{\mathsf{op}}) \otimes_{H^{\mathsf{e}}} (Q(H) \otimes Q(H^{\mathsf{op}})))\\
&\qquad \cong \operatorname{RHom}_{H^{\mathsf{e}}}(H, {}^{\nu}Q(H)^1 \otimes {}^{\mu}Q(H^{\mathsf{op}})^1[2d])\\
&\qquad \cong \operatorname{RHom}_{Q(H)^{\mathsf{e}}}(Q(H), {}^{\nu}Q(H)^1 \otimes {}^{\mu}Q(H^{\mathsf{op}})^1[2d])
\end{aligned}$$

for some automorphism ν and μ of $Q(H)$ and $Q(H^{\mathsf{op}})$ respectively, where the first isomorphism follows from [YZ3, Lemma 3.7(1)], the second one follows from the fact $R = {}^{\nu}H^1[d]$, and the third one follows from [YZ3, Lemma 3.7(2)].

By the rigidity of the dualizing complex R,

$$\begin{aligned}\mathrm{RHom}_{H^{\mathsf{e}}}(H, R\otimes R^{\mathsf{op}}) &\otimes_{H^{\mathsf{e}}} (Q(H)\otimes Q(H^{\mathsf{op}}))\\ &\cong R\otimes_{H^{\mathsf{e}}} (Q(H)\otimes Q(H^{\mathsf{op}}))\\ &\cong {}^{\nu}H^1[d]\otimes_{H^{\mathsf{e}}} (Q(H)\otimes Q(H^{\mathsf{op}}))\\ &\cong {}^{\nu}Q(H)^1[d].\end{aligned}$$

Combining these two isomorphisms and after twisting by some automorphisms, we obtain that

$$\mathrm{RHom}_{Q(H)^{\mathsf{e}}}(Q(H), Q(H)\otimes Q(H^{\mathsf{op}}))\cong {}^{\tau}Q(H)^1[-d]$$

for some automorphism τ of $Q(H)$. Using the decomposition of $Q(H)\otimes Q(H^{\mathsf{op}})$ induced by (E3.2.1), we have

$$\mathrm{RHom}_{D_j^{\mathsf{e}}}(D_j, D_j\otimes D_j^{\mathsf{op}})\cong {}^{\tau_j}D_j^1[-d] \tag{E3.2.2}$$

for all j. Since D_j is smooth, $\mathrm{projdim}_{D_j^{\mathsf{e}}} D_j<\infty$. Recall that H is a compact object in the derived category $\mathsf{D}(H^{\mathsf{e}}\text{-}\mathsf{Mod})$. This implies that $Q(H)$ is a compact object in $\mathsf{D}(Q(H)^{\mathsf{e}}\text{-}\mathsf{Mod})$. Hence D_j is a compact object in $\mathsf{D}((D_j)^{\mathsf{e}}\text{-}\mathsf{Mod})$. Then

$$\mathrm{projdim}_{D_j^{\mathsf{e}}} D_j=\min\{i\,:\,\mathrm{Ext}^i_{D_j^{\mathsf{e}}}(D_j, D_j^{\mathsf{e}})\neq 0\}=d.$$

By [YZ5, Lemma 1.3(b)], $\mathrm{gldim}\, D_j^{\mathsf{e}}=\mathrm{projdim}_{D_j^{\mathsf{e}}} D_j=d$. Clearly,

$$\mathrm{injdim}\, D_j^{\mathsf{e}}\leq \mathrm{gldim}\, D_j^{\mathsf{e}}=d \tag{E3.2.3}$$

and

$$\mathrm{injdim}\, D_j^{\mathsf{e}}\geq \min\{i\,:\,\mathrm{Ext}^i_{D_j^{\mathsf{e}}}(D_j, D_j^{\mathsf{e}})\neq 0\}=d. \tag{E3.2.4}$$

Combining (E3.2.2), (E3.2.3) and (E3.2.4), we see that D_j is smooth, rigid of homological transcendence degree d for every j. ∎

We have an immediate consequence.

Corollary 3.3: (a) *Suppose* char $k=p\geq 0$. *Let G be a polycyclic-by-finite group containing no elements of order p. Let $Q(kG)=\bigoplus_j D_j$ where each D_j is a simple artinian algebra. Then D_j is smooth and rigid and* $\mathrm{Htr}\, D_j=d$ *for all j, where d is the Hirsch length of G.*

(b) *Let $\mathfrak{g}$ be a finite dimensional Lie algebra. Then $Q(U(\mathfrak{g}))$ is smooth and rigid of* $\mathrm{Htr}=\dim\mathfrak{g}$.

(c) *Let $\mathcal{O}_q(G)$ be the standard generic quantized coordinate ring of a connected semisimple Lie group G (see* [BZ, Section 6.5] *for details). Then $Q(\mathcal{O}_q(G))$ is smooth and rigid of* $\mathrm{Htr} = \dim G$.

(d) *Let $U_q(\mathfrak{g})$ be the quantized enveloping algebra of the semisimple Lie algebra $\mathfrak{g}$ (see* [BZ, Section 6.4] *for details). Then $Q(U_q(\mathfrak{g}))$ is smooth and rigid of* $\mathrm{Htr} = \dim \mathfrak{g}$.

Remark 3.4: Corollary 3.3(a,b) is a slight generalization of a result of Stafford [St, Proposition 1.7]. Corollary 3.3(b) is also given in [YZ5, Example 1.9(f)]. Corollary 3.3(d) can be viewed as a quantum version of Corollary 3.3(b).

Next we consider a partial converse of Theorem 3.2.

LEMMA 3.5: *Let H be a noetherian Hopf algebra that is prime as an algebra. Suppose that the simple artinian ring $Q(H)$ is rigid of* $\mathrm{Htr} = d$. *Then*

$$\mathrm{Ext}^i_H(k,H) = \begin{cases} k & i = d, \\ 0 & i \neq d. \end{cases}$$

Proof. A computation similar to the one in the proof of Theorem 3.2 shows

$$\mathrm{RHom}_{H^{\mathsf{e}}}(H, H^{\mathsf{e}}) \otimes_{H^{\mathsf{e}}} Q(H)^{\mathsf{e}} \cong \mathrm{RHom}_{Q(H)^{\mathsf{e}}}(Q(H), Q(H)^{\mathsf{e}}).$$

Since $Q(H)$ is rigid of $\mathrm{Htr} = d$, $\mathrm{RHom}_{Q(H)^{\mathsf{e}}}(Q(H), Q(H)^{\mathsf{e}}) \cong {}^{\mu}Q(H)^1[-d]$ for some automorphism μ of $Q(H)$. Hence

$$\mathrm{Ext}^i_{H^{\mathsf{e}}}(H, H^{\mathsf{e}}) \otimes_{H^{\mathsf{e}}} Q(H)^{\mathsf{e}} = \begin{cases} {}^{\mu}Q(H)^1 & i = d, \\ 0 & i \neq d. \end{cases}$$

By Lemmas 2.1(c) and 2.2, there are right H^{op}-module isomorphisms

$$\begin{aligned} \mathrm{Ext}^i_{H^{\mathsf{e}}}(H, H^{\mathsf{e}}) &\cong \mathrm{Ext}^i_H(k, L(H^{\mathsf{e}})) \cong \mathrm{Ext}^i_H(k, {}_H H \otimes H^{\mathsf{op}}) \\ &\cong \mathrm{Ext}^i_H(k, H) \otimes H^{\mathsf{op}}. \end{aligned}$$

So $\mathrm{Ext}^i_{H^{\mathsf{e}}}(H, H^{\mathsf{e}})$ is either zero or a free right H^{op}-module. A right H^{op}-module can be viewed as a left H-module. So any nonzero $\mathrm{Ext}^i_{H^{\mathsf{e}}}(H, H^{\mathsf{e}})$ is a free left H-module. By symmetry, it is a free right H-module when nonzero. If $\mathrm{Ext}^i_{H^{\mathsf{e}}}(H, H^{\mathsf{e}})$ is nonzero, then $\mathrm{Ext}^i_{H^{\mathsf{e}}}(H, H^{\mathsf{e}}) \otimes_{H^{\mathsf{e}}} Q(H)^{\mathsf{e}}$ is nonzero, which implies that $i = d$. In this case, the rank of the left (and the right) free H-module $\mathrm{Ext}^d_{H^{\mathsf{e}}}(H, H^{\mathsf{e}})$ must be one. Finally the assertion that $\mathrm{Ext}^d_H(k, H) \cong k$ follows from the isomorphism $\mathrm{Ext}^d_{H^{\mathsf{e}}}(H, H^{\mathsf{e}}) \cong \mathrm{Ext}^d_H(k, H) \otimes H^{op}$. ∎

Proposition 3.6: *Suppose H is a noetherian prime Hopf algebra of finite global dimension. If $Q(H)$ is a rigid simple artinian ring, then H is AS-regular.*

Proof. By Lemma 3.5

$$\mathrm{Ext}^i_H(k,H)=\begin{cases} k & i=d,\\ 0 & i\neq d,\end{cases}$$

where $d=\mathrm{Htr}\,Q(H)$. Since H has finite global dimension, by [LL, Corollary 2.4]

$$\mathrm{gldim}\,H=\mathrm{projdim}_H\,k=d.$$

Therefore, H is AS regular. ∎

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